

AXIOMATIC CLOSURE OPERATIONS, PHANTOM EXTENSIONS, AND SOLIDITY

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ABSTRACT. In this article, we generalize a previously defined set of axioms for a closure operation that induces balanced big Cohen-Macaulay modules. While the original axioms were only defined in terms of finitely generated modules, these new ones will apply to all modules over a local domain. The new axioms will lead to a notion of phantom extensions for general modules, and we will prove that all modules that are phantom extensions can be modified into balanced big Cohen-Macaulay modules and are also solid modules. As a corollary, if R has characteristic $p > 0$ and is F -finite, then all solid algebras are phantom extensions. If R also has a big test element (e.g., if R is complete), then solid algebras can be modified into balanced big Cohen-Macaulay modules. (Hochster and Huneke have previously demonstrated that there exist solid algebras that cannot be modified into balanced big Cohen-Macaulay algebras.) We also point out that tight closure over local domains in characteristic p generally satisfies the new axioms and that the existence of a big Cohen-Macaulay module induces a closure operation satisfying the new axioms.

Let R be a local domain. In [D10], we presented seven axioms for a closure operation that were necessary and sufficient in order to prove the existence of balanced big Cohen-Macaulay modules over R . (Recall that a balanced big Cohen-Macaulay module B over R is a module for which every partial system of parameters for R is a regular sequence on B .) The work is independent of characteristic considerations for R , but whether or not there exists such a closure operation satisfying the axioms is in general a major open problem. Tight closure in equal characteristic does satisfy these axioms, and the existence of big Cohen-Macaulay algebras over R implies the existence of such a closure operation in mixed characteristic for rings of Krull dimension at most 3 [He02, Ho02].

In the present article, we will generalize the seven axioms to include closure within general R -modules, not just finitely generated modules. The axioms of [D10] will be termed *finite axioms* while these new axioms will be called the *big axioms*. We will show that tight closure in characteristic p satisfies the big axioms (Proposition 1.6) and that the existence of a balanced big Cohen-Macaulay R -module induces a closure operation on modules that satisfies the big axioms (Proposition 1.4). Given a closure operation satisfying the finite axioms, we prove (Proposition 1.5) that this operation can be extended to a closure for all R -modules that satisfies the big axioms. The technique is based on the idea of finitistic tight closure in [HH90, Definition 8.19].

In the second section, we will use the big axioms to define when a general R -module M is a phantom extension of R with respect to the closure operation.

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Although this definition appears to depend upon choices of a free presentation and lifts of certain maps, we prove that it is in fact independent of all choices. In the following section, we will prove that all phantom extensions can be modified into balanced big Cohen-Macaulay R -modules (Theorem 3.7). In the fourth section, we use the earlier results to prove that if $\alpha : R \rightarrow M$ is phantom over a complete, local domain, then M is a solid module. We also prove that in prime characteristic p over an F -finite ring R , solid algebras are phantom extensions (Theorem 4.8). The resulting Corollary 4.9 is that solid algebras can be modified into big Cohen-Macaulay modules, which complements the example of Hochster and Huneke [HH09, Example 3.14] of a solid algebra in characteristic p that cannot be modified into a big Cohen-Macaulay algebra. On the other hand, we show that there are solid modules that cannot be phantom extensions.

1. BIG AND FINITE CLOSURE AXIOMS

We will alter the notation of [D10] (e.g., N_M^\natural) and adopt some of the notation and naming conventions used in [E12] to emphasize the generic, axiomatic nature of our closure operations and to avoid confusion with other closure operations that may use the \natural symbol as well.

Definition 1.1. Let (R, \mathfrak{m}) be a local domain. All modules named below are arbitrary and not necessarily finitely generated. The following is a list of axioms that may be satisfied by an operation cl on submodules N of M .

- (1) (extension) N_M^{cl} is a submodule of M containing N .
- (2) (idempotence) $(N_M^{cl})_M^{cl} = N_M^{cl}$.
- (3) (order-preserving) If $N \subseteq M \subseteq W$, then $N_W^{cl} \subseteq M_W^{cl}$.
- (4) (functorial) Let $f : M \rightarrow W$ be a homomorphism. Then $f(N_M^{cl}) \subseteq f(N)_W^{cl}$.
- (5) (semi-residual) If $N_M^{cl} = N$, then $0_{M/N}^{cl} = 0$.
- (6) The ideal \mathfrak{m} is closed; i.e., $\mathfrak{m}_R^{cl} = \mathfrak{m}$.
- (7) (generalized colon-capturing) Let x_1, \dots, x_{k+1} be a partial system of parameters for R , and let $J = (x_1, \dots, x_k)$. Suppose that there exists a surjective homomorphism $f : M \rightarrow R/J$ and $v \in M$ such that $f(v) = x_{k+1} + J$. Then

$$(Rv)_M^{cl} \cap \ker f \subseteq (Jv)_M^{cl}.$$

We will refer to the list above as the *big axioms*. If the closure operation only satisfies the axioms for finitely generated modules, then we say that the operation meets the *finite axioms*.

Remark 1.2. Note that we removed the condition $0_R^{cl} = 0$ from the sixth axiom. A close reading of [D10] shows that the only citation of that axiom was in Lemma 3.12, but changing the equality at the end of the displayed formula to an inclusion \supseteq still yields the result but does so using axiom (1) instead of $0_R^{cl} = 0$.

It is worth pointing out that the results of [D10, Lemmas 1.2 and 1.3, Proposition 1.4] concerning quotients, direct sums, intersections, sums, and colon-capturing are still valid for an operation that satisfies the big axioms. The proofs are identical as they make no reference to finite generation of modules. We restate a few for ease of reference and add to that list a result related to isomorphisms. Additionally, Mel Hochster and Rebecca R.G. pointed out that the condition $0_R^{cl} = 0$ follows from the other axioms.

Lemma 1.3. *Let cl be an operation satisfying the big (or finite) axioms. All R -modules are arbitrary (or finite).*

- (a) [D10, Lemma 1.2(a)] *Let $N' \subseteq N \subseteq M$ be R -modules. Then $u \in N_M^{cl}$ if and only if $u + N' \in (N/N')_{M/N'}^{cl}$.*
- (b) [D10, Lemma 1.2(b)] *If $N_i \subseteq M_i$, then $(N_1 \oplus N_2)_{M_1 \oplus M_2}^{cl} = (N_1)_{M_1}^{cl} \oplus (N_2)_{M_2}^{cl}$.*
- (c) [D10, Lemma 1.3] *Let x_1, \dots, x_{k+1} be a partial system of parameters for R , and let $J = (x_1, \dots, x_k)$. Suppose that there exists a homomorphism $f : M \rightarrow R/J$ and $v \in M$ such that $f(v) = x_{k+1} + J$. Then $(Rv)_M^{cl} \cap \ker f \subseteq (Jv)_M^{cl}$.*
- (d) *If $f : M \rightarrow M'$ is an isomorphism of R -modules with $N \subseteq M$ and $N' = f(N)$, then $u \in N_M^{cl}$ if and only if $f(u) \in (N')_{M'}^{cl}$.*
- (e) $0_R^{cl} = 0$.

Proof. For (a), (b), and (c), use the existing proofs in [D10] as finiteness of modules is irrelevant to the proofs. For (d), apply axiom (4) to f and to f^{-1} to get the two containments.

¹For (e), first note that if R is a local domain of Krull dimension 0, then $0 = \mathfrak{m}$ and so is closed by axiom (6). If R has Krull dimension greater than 0, then suppose that $x \in 0_R^{cl} \setminus 0$. We can consider $x = x_1$ a partial s.o.p., and let $J = (0)$ be the ideal generated by the “other” parameters in the partial system. Let $M = R \oplus R$ and define $f : M \rightarrow R = R/J$ by $f(r, s) = rx + s$ so that $v = (1, 0) \mapsto x_1$. Then $f(-1, x) = 0$, and $(-1, x) \in (Rv)_M^{cl}$ as $(Rv)_M^{cl} = (R \oplus 0)_{R \oplus R}^{cl} = R_R^{cl} \oplus 0_R^{cl} = R \oplus 0_R^{cl}$, which contains $(-1, 0)$ and $(0, x)$. Hence, axioms (6) and (7) imply that $(-1, x) \in (Jv)_M^{cl} = 0_M^{cl} = 0_R^{cl} \oplus 0_R^{cl} \subseteq \mathfrak{m} \oplus R$, which is impossible. \square

The axioms and theorems developed in [D10] are results of the finite axioms. In this paper we will develop analogues and new results for the big axioms. Although the big axioms may appear more powerful (or more restrictive) at first compared to the finite axioms, they are equivalent to each other in the following sense.

Proposition 1.4. *Let (R, \mathfrak{m}) be a local domain. The following are equivalent:*

- (i) *There exists a closure operation c that satisfies the big axioms (1)–(7).*
- (ii) *There exists a closure operation d that satisfies the finite axioms (1)–(7).*
- (iii) *There exists a balanced big Cohen-Macaulay module B over R .*

Proof. (i) implies (ii) is obvious with $d = c$. (ii) implies (iii) is the main result (Theorem 3.16) of [D10]. To see (iii) implies (i), we just note that the proof of Theorem 4.2 in [D10] suffices as finite generation of modules is irrelevant to the arguments, even though the theorem is stated to prove the existence of an operation meeting the finite axioms. \square

Although the proposition above implies that a closure operation d meeting the finite axioms implies the existence of an operation c meeting the big axioms, there is nothing guaranteeing that c and d agree for finitely generated modules. On the other hand, an operation satisfying the finite axioms can be extended to one meeting the big axioms using the idea of finitistic tight closure given in [HH90, Definition 8.19].

Proposition 1.5. *Let (R, \mathfrak{m}) be a local domain. Suppose that cl is an operation on modules that satisfies the finite axioms. Then cl can be extended to all modules so that the extended operation satisfies the big axioms.*

¹The proof of this result comes from Mel Hochster and Rebecca R.G.

Proof. Given a notion of N_M^{cl} for finitely generated modules $N \subseteq M$ that satisfies the finite axioms, we extend cl to all modules $N \subseteq M$ via $u \in N_M^{cl}$ if and only if there exists a finitely generated submodule $M_0 \subseteq M$ such that $u \in (N \cap M_0)_{M_0}^{cl}$. This definition leaves N_M^{cl} unchanged for finitely generated modules M due to the inclusion $\iota : M_0 \rightarrow M$, the containments $N \cap M_0 \subseteq N \subseteq M$, and finite axioms (4) and (3). Thus, this definition is an extension of cl to arbitrary modules.

Let $N \subseteq M$ be arbitrary modules. We check that the big axioms hold.

- (1) If $u \in N$, then $u \in (N \cap Ru)_{Ru}^{cl}$ by finite axiom (1) so that $N \subseteq N_M^{cl}$. Let $r \in R$ and $u_1, u_2 \in N_M^{cl}$. Then there exist finitely generated submodules M_1, M_2 of M such that $u_i \in (N \cap M_i)_{M_i}^{cl}$. Let $M_0 = M_1 + M_2$, a finitely generated submodule of M . Then finite axioms (3) and (4) imply that $u_i \in (N \cap M_0)_{M_0}^{cl}$. Thus, finite axiom (1) implies that $ru_1 + u_2 \in (N \cap M_0)_{M_0}^{cl}$ and so $ru_1 + u_2 \in N_M^{cl}$.
- (2) Suppose that $u \in (N_M^{cl})_M^{cl}$. We claim that $u \in N_M^{cl}$. There exists a finitely generated submodule $M_0 \subseteq M$ such that $u \in (N_M^{cl} \cap M_0)_{M_0}^{cl}$. Let w_1, \dots, w_k be generators of the finitely generated submodule $N_M^{cl} \cap M_0$ of M_0 . Then for each j there exists a finitely generated submodule M_j of M such that $w_j \in (N \cap M_j)_{M_j}^{cl}$ as each w_j is in N_M^{cl} . Let $Q = M_0 + M_1 + \dots + M_k$, a finitely generated submodule of M . By finite axioms (4) and (3), $w_j \in (N \cap M_j)_{M_j}^{cl} \subseteq (N \cap M_j)_Q^{cl} \subseteq (N \cap Q)_Q^{cl}$ for each j , and so $N_M^{cl} \cap M_0 \subseteq (N \cap Q)_Q^{cl}$. Thus, finite axioms (4), (3), and (2) imply that

$$(N_M^{cl} \cap M_0)_{M_0}^{cl} \subseteq (N_M^{cl} \cap M_0)_Q^{cl} \subseteq ((N \cap Q)_Q^{cl})_Q^{cl} = (N \cap Q)_Q^{cl}$$

which means that $u \in (N \cap Q)_Q^{cl}$ and so $u \in N_M^{cl}$ as claimed.

- (3) follows from finite axioms (3) as $(N \cap W_0)_{W_0}^{cl} \subseteq (M \cap W_0)_{W_0}^{cl}$ for any finitely generated submodule W_0 of W .
- (4) follows from finite axioms (3) and (4) as M_0 finitely generated in M implies that $f(M_0)$ is a finitely generated submodule of W and $f(N \cap M_0) \subseteq f(N) \cap f(M_0)$.
- (5) Suppose that $N_M^{cl} = N$ and $\bar{u} \in 0_{M/N}^{cl}$ where $u \in M$. Then there exists a finitely generated submodule M_0 of M such that $\bar{u} \in 0_{\frac{M_0+N}{N}}^{cl}$. Thus big axiom (4) and Lemma 1.3(d) imply that $\bar{u} \in 0_{\frac{M_0}{M_0 \cap N}}^{cl}$. Then Lemma 1.3(a) applied to the finite module $\frac{M_0}{M_0 \cap N}$ implies that $u \in (N \cap M_0)_{M_0}^{cl}$ and so $u \in N_M^{cl} = N$.
- (6) obvious
- (7) Let x_1, \dots, x_{k+1} be a partial system of parameters, $J = (x_1, \dots, x_k)$, M an arbitrary R -module, $f : M \rightarrow R/J$ with $v \in M$ such that $f(v) = \overline{x_{k+1}}$. Suppose that $u \in (Rv)_M^{cl}$ and $f(u) = 0$. Then there exists a finite module $M_0 \subseteq M$ such that $u \in (Rv \cap M_0)_{M_0}^{cl}$. Using finite axioms (3) and (4), we can assume without loss of generality that $v \in M_0$ and so $u \in (Rv)_{M_0}^{cl}$. Restricting f to M_0 and applying Lemma 1.3(c) shows that $u \in (Jv)_{M_0}^{cl}$ and so $u \in (Jv)_M^{cl}$.

□

We also note that tight closure in characteristic p defined for arbitrary modules satisfies the big axioms, but the proof requires the use of a big test element in order to prove idempotence.

Let (R, \mathfrak{m}) be a local domain of prime characteristic p . If F^e represents the iterated Frobenius homomorphism $r \mapsto r^{p^e} = r^q$, then let $F^e(M)$ denote $F^e(R) \otimes_R M$ for an arbitrary R -module M and let $N_M^{[q]} = \text{Im}(F^e(N) \rightarrow F^e(M))$. Then $u \in N_M^*$ if and only if there exists $c \neq 0$ such that $c \otimes u = cu^q \in N_M^{[q]}$ for all $e \geq 0$. See [HH90, Section 8] or [Ho07] for further details on tight closure of modules. We say that R possesses a big test element c if the same c can be used in all tight closure calculations for all pairs of modules $N \subseteq M$. Big test elements are known to exist in a large number of cases, e.g., if R is also complete. See [Ho07, p. 150]. Note that we are not using finitistic tight closure and that whether these two concepts are the same or not is still an open problem.

Proposition 1.6. *Let (R, \mathfrak{m}) be a local domain of prime characteristic p that possesses a big test element, e.g., R is also complete. Then tight closure defined as above for all R -modules obeys the big axioms.*

Proof. Let $N \subseteq M$ be arbitrary R -modules.

- (1) Proposition 8.5(a) of [HH90]
- (2) Let c be a big test element for R . If $u \in (N_M^*)^*$, then $cu^q \in (N_M^*)^{[q]}$. Thus, $cu^q = r_1 u_1^q + \cdots + r_n u_n^q$, where $r_i \in R$ and $u_i \in N_M^*$. Then $cu_i^q \in N_M^{[q]}$, and $c^2 u^q \in N_M^{[q]}$ for all q , which implies that $u \in N_M^*$.
- (3) Proposition 8.5(b) of [HH90]
- (4) Lemma 5.5 of [D10] gives a valid proof, even though the statement unnecessarily presumes the modules are finitely generated.
- (5) Remark 8.4 in [HH90]
- (6) \mathfrak{m} is a prime ideal, which is always tightly closed.
- (7) The proof of Proposition 5.6 of [D10] still applies when M is not necessarily finite.

□

2. PHANTOM EXTENSIONS

In this section we will generalize and refine the notion of phantom extensions introduced in [HH94, Section 5] with respect to tight closure for prime characteristic rings and generalized in [D10] to any closure operation obeying the finite axioms independent of characteristic. We expand the notion to arbitrary R -modules and remove references to matrix representations of maps. In the end, the description below is equivalent to the other two in the cases of characteristic p or finite modules. Throughout this section (R, \mathfrak{m}) is a local domain.

Discussion 2.1. Let M be an arbitrary R -module along with an injection $\alpha : R \rightarrow M$ yielding the short exact sequence $0 \rightarrow R \xrightarrow{\alpha} M \xrightarrow{\beta} Q \rightarrow 0$. Let $G \xrightarrow{\nu} F \xrightarrow{\mu} Q \rightarrow 0$ be any free presentation of Q . Choose a lifting $\tilde{\mu} : F \rightarrow M$ of μ such that $\mu = \beta \circ \tilde{\mu}$.

Consider the map $\tilde{\mu} \circ \nu : G \rightarrow M$. Then $\beta \circ (\tilde{\mu} \circ \nu) = \mu \circ \nu = 0$, and so $\tilde{\mu} \circ \nu$ maps G into the $\ker \beta = \text{Im } \alpha$. Hence $\tilde{\mu} \circ \nu : G \rightarrow \text{Im } \alpha \subseteq M$. Since α is injective, we have a map $\tilde{\nu} = \alpha^{-1} \circ \tilde{\mu} \circ \nu : G \rightarrow R$. All told, we have a commutative diagram

with exact rows:

$$(2.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & R & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & Q \longrightarrow 0 \\ & & \uparrow \tilde{\nu} & & \uparrow \tilde{\mu} & & \uparrow \text{id} \\ & & G & \xrightarrow{\nu} & F & \xrightarrow{\mu} & Q \longrightarrow 0 \end{array}$$

Note that given the injection $\alpha : R \rightarrow M$, we made only two choices in constructing the diagram above: the choice of a free presentation and the choice of the lifting $\tilde{\mu}$; the map $\tilde{\nu}$ was uniquely determined by the other choices.

Let $(-)^{\vee}$ denote the dual operator $\text{Hom}_R(-, R)$. Then $\nu^{\vee} : F^{\vee} \rightarrow G^{\vee}$ is the map sending $h \in F^{\vee}$ to $h \circ \nu \in G^{\vee}$, and $\tilde{\nu} \in G^{\vee}$.

Definition 2.3. Let cl be an operation that obeys the big axioms. Given an injection $\alpha : R \rightarrow M$ and a diagram as in (2.2), we say that M is a *phantom extension of R via α* if $\tilde{\nu} \in (\text{Im } \nu^{\vee})_{G^{\vee}}^{cl}$.

Remark 2.4. This definition is equivalent in characteristic p for finite modules to that given in [HH94, Definition 5.2] due to Theorem 5.13 of the same article. It is also equivalent to the definition given in [D10, Definition 2.2] with a different choice for notation. Essentially, to say that α is a phantom extension means that the cycle in $\text{Ext}_R^1(Q, R)$ that corresponds to the SES built from α also lives in the closure of the boundaries. We have chosen, however, not to refer to membership in the module $\text{Ext}_R^1(Q, R)$ but leave everything in terms of diagrams instead.

The definition is independent of both the choice of presentation and the lift $\tilde{\mu}$.

Lemma 2.5. Let cl be an operation obeying the big axioms, and let $\alpha : R \rightarrow M$ be an injection. If $\tilde{\nu} \in (\text{Im } \nu^{\vee})_{G^{\vee}}^{cl}$ for some choice of free presentation and some choice of $\tilde{\mu}$, then it is true for all free presentations and choices of $\tilde{\mu}$.

Proof. First suppose that a choice of free presentation has been made, but two choices of lifts $\tilde{\mu}$ and $\tilde{\mu}'$ have been made, along with corresponding $\tilde{\nu}$ and $\tilde{\nu}'$.

$$\begin{array}{ccccccc} R & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & Q & \longrightarrow & 0 \\ \uparrow \tilde{\nu} & & \uparrow \tilde{\mu} & & \uparrow \text{id} & & \\ \uparrow \tilde{\nu}' & & \uparrow \tilde{\mu}' & & & & \\ G & \xrightarrow{\nu} & F & \xrightarrow{\mu} & Q & \longrightarrow & 0 \end{array}$$

Since $\beta \circ \tilde{\mu} = \mu = \beta \circ \tilde{\mu}'$, we have $\tilde{\mu} - \tilde{\mu}' : F \rightarrow \ker \beta = \text{Im } \alpha \subseteq M$ and so $\alpha^{-1} \circ (\tilde{\mu} - \tilde{\mu}') : F \rightarrow R$ is a well-defined map. Moreover,

$$\nu^{\vee}(\alpha^{-1} \circ (\tilde{\mu} - \tilde{\mu}')) = \alpha^{-1} \circ (\tilde{\mu} - \tilde{\mu}') \circ \nu = \tilde{\nu} - \tilde{\nu}'$$

so that $\tilde{\nu} - \tilde{\nu}' \in \text{Im } \nu^{\vee} \subseteq (\text{Im } \nu^{\vee})_{G^{\vee}}^{cl}$ by axiom (1). Therefore, $\tilde{\nu} \in (\text{Im } \nu^{\vee})_{G^{\vee}}^{cl}$ if and only if $\tilde{\nu}' \in (\text{Im } \nu^{\vee})_{G^{\vee}}^{cl}$.

Now suppose that $\tilde{\nu} \in (\text{Im } \nu^{\vee})_{G^{\vee}}^{cl}$ for a certain diagram (2.2) and that $G' \xrightarrow{\nu'} F' \xrightarrow{\mu'} Q \rightarrow 0$ is another free presentation of Q . Since G' and F' are free, we can

build a commutative diagram

$$\begin{array}{ccccccc}
 R & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & Q & \longrightarrow & 0 \\
 \uparrow \tilde{\nu} & & \uparrow \tilde{\mu} & & \uparrow \text{id} & & \\
 G & \xrightarrow{\nu} & F & \xrightarrow{\mu} & Q & \longrightarrow & 0 \\
 \uparrow \phi & & \uparrow \psi & & \uparrow \text{id} & & \\
 G' & \xrightarrow{\nu'} & F' & \xrightarrow{\mu'} & Q & \longrightarrow & 0
 \end{array}$$

Let $\tilde{\mu}' = \tilde{\mu} \circ \psi$ and let $\tilde{\nu}' = \tilde{\nu} \circ \phi$. Consider the dual map $\phi^\vee : G^\vee \rightarrow (G')^\vee$, where $\tilde{\nu}' = \phi^\vee(\tilde{\nu})$ and $\phi^\vee(\text{Im } \nu^\vee) \subseteq \text{Im } (\nu')^\vee$ as

$$\phi^\vee \circ \nu^\vee(h) = h \circ \nu \circ \phi = h \circ \psi \circ \nu' = (\nu')^\vee(h \circ \psi)$$

for all $h : F \rightarrow R$. Therefore, axioms (3) and (4) imply that $\tilde{\nu}' \in (\text{Im } (\nu')^\vee)_{(G')^\vee}^{cl}$. Since this latter condition is true for this particular $\tilde{\mu}'$ and $\tilde{\nu}'$, then it is true for all choices of such maps using the free presentation $G' \xrightarrow{\nu'} F' \xrightarrow{\mu'} Q \rightarrow 0$ according to the first part of the proof. \square

Notice that if $\alpha : R \rightarrow R$ is the identity map, then $Q = 0$ and one can choose $G = F = 0$ and $\tilde{\nu} = 0$ -map. It is thus trivially true that $\tilde{\nu} \in (\text{Im } \nu^\vee)_{G^\vee}^{cl}$ and so R is trivially a phantom extension of itself via the identity map.

Discussion 2.6. Given an injection $\alpha : R \rightarrow M$, expand diagram (2.2) to

$$\begin{array}{ccccccc}
 R & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & Q & \longrightarrow & 0 \\
 \uparrow \tilde{\nu} & & \uparrow \tilde{\mu} & & \uparrow \text{id} & & \\
 G & \xrightarrow{\nu} & F & \xrightarrow{\mu} & Q & \longrightarrow & 0 \\
 \uparrow \text{id} & & \uparrow \pi & & \uparrow \beta & & \\
 G & \xrightarrow{\nu_1} & F \oplus R & \xrightarrow{\mu_1} & M & \longrightarrow & 0
 \end{array}$$

where $\pi(f, r) = f$, $\mu_1(f, r) = \tilde{\mu}(f) - \alpha(r)$, and $\nu_1(g) = (\nu(g), \tilde{\nu}(g))$.

The diagram commutes because $\pi \circ \nu_1(g) = \nu(g) = \nu \circ \text{id}(g)$, and

$$\beta \circ \mu_1(f, r) = \beta \circ \tilde{\mu}(f) - \beta \circ \alpha(r) = \mu(f) - 0 = \mu \circ \pi(f, r).$$

The bottom row is also exact. The map μ_1 is surjective as a given $m \in M$ leads to $\beta(m) = \mu(f) \in Q$, and $\beta(m - \tilde{\mu}(f)) = \mu(f) - \mu(f) = 0$ leads to $m - \tilde{\mu}(f) = \alpha(r)$ for an $r \in R$. Thus, $m = \tilde{\mu}(f) + \alpha(r) = \mu_1(f, -r)$. Also we have exactness in the middle. First, $\mu_1 \circ \nu_1(g) = \tilde{\mu}(\nu(g)) - \alpha(\tilde{\nu}(g)) = 0$. Second, if $\mu_1(f, r) = 0$, then $\tilde{\mu}(f) = \alpha(r)$ and so $\mu(f) = \beta(\tilde{\mu}(f)) = \beta(\alpha(r)) = 0$. Hence, $f = \nu(g)$ for some $g \in G$, and $\tilde{\nu}(g) = \alpha^{-1} \circ \tilde{\mu} \circ \nu(g) = \alpha^{-1}(\tilde{\mu}(f)) = \alpha^{-1}(\alpha(r)) = r$. Thus, $\nu_1(g) = (\nu(g), \tilde{\nu}(g)) = (f, r)$, and so the bottom row is exact.

We have proven the following.

Lemma 2.8. *Given an injection $\alpha : R \rightarrow M$, one can build a commutative diagram (2.7) with exact rows.*

This diagram will allow us to understand $\alpha(1)$ within M better.

Lemma 2.9. *Let cl be an operation satisfying the big axioms. Let $\alpha : R \rightarrow M$ be an injection with associated diagram (2.7). If $\alpha(1) \in \mathfrak{m}M$, then $\tilde{\nu}$ is onto. The converse is true when $\text{Im } \nu \subseteq \mathfrak{m}F$, e.g., the free presentation is minimal.*

Proof. Suppose that $\alpha(1) \in \mathfrak{m}M$. Then $\alpha(1) \in \mathfrak{m}\text{Im } \tilde{\mu} + \mathfrak{m}\text{Im } \alpha \subseteq \text{Im } \tilde{\mu} + \mathfrak{m}\alpha(1)$, which implies that $\alpha(1) \in \text{Im } \tilde{\mu}$, i.e., $\alpha(1) = \tilde{\mu}(f)$ for some $f \in F$. Thus, $\mu_1(f, 1) = \tilde{\mu}(f) - \alpha(1) = 0$. By exactness, $(f, 1) = \nu_1(g)$ for some $g \in G$, and so $\tilde{\nu}(g) = 1$, making $\tilde{\nu}$ surjective.

Now suppose that $\tilde{\nu}(g) = 1$ for some $g \in G$ and that $\text{Im } \nu \subseteq \mathfrak{m}F$. Then $0 = \mu_1 \circ \nu_1(g) = \mu_1(\nu(g), 1) = \tilde{\mu}(\nu(g)) - \alpha(1)$ and so $\alpha(1) \in \tilde{\mu}(\text{Im } \nu) \subseteq \tilde{\mu}(\mathfrak{m}F) \subseteq \mathfrak{m}M$. \square

Corollary 2.10. *Let cl be an operation satisfying the big axioms. If $\alpha : R \rightarrow M$ is phantom, then $\alpha(1) \notin \mathfrak{m}M$.*

Proof. Given α , build a diagram (2.2) using a minimal free presentation so that $\tilde{\nu} \in (\text{Im } \nu^\vee)_{G^\vee}^{\text{cl}}$. (We can do this by Lemma 2.5.) For a contradiction, suppose that $\tilde{\nu}(g^*) = 1$ for some $g^* \in G$. Let $\phi : G^\vee \rightarrow R$ be the evaluation map $\phi(h) = h(g^*)$. Then $\phi(\tilde{\nu}) = 1$. If $\eta \in F^\vee$, then $\phi(\eta \circ \nu) = \eta(\nu(g^*)) \in \mathfrak{m}$ as $\text{Im } \nu \subseteq \mathfrak{m}F$. Thus, $\phi(\text{Im } \nu^\vee) \subseteq \mathfrak{m}$. Axioms (3), (4), and (6) imply that $1 = \phi(\tilde{\nu}) \in \mathfrak{m}_R^{\text{cl}} = \mathfrak{m}$, a contradiction. \square

3. PHANTOM EXTENSIONS AND BIG COHEN-MACAULAY MODULES

In this section, we show that phantom extensions can be modified into balanced big Cohen-Macaulay modules. This was also done in [D10, Section 3] but only for finitely-generated phantom extensions. We now extend the result to arbitrary phantom extensions and refine the argument to avoid the use of matrices and basis choices. As in [Ho75], [HH94], and [D10], we will make use of module modifications in order to construct a balanced big Cohen-Macaulay module by forcing relations on systems of parameters to be trivialized in a sequence of modules. The key part is to show that one prevents the limit of the sequence from becoming the zero module. The first step for us though is to build a commutative diagram like (2.7) with some extra properties based on a relation on a system of parameters in the module M .

Discussion 3.1. Let M be an arbitrary R -module with $\alpha : R \rightarrow M$ an injection. Let $x_1 \dots, x_{k+1}$ be part of a system of parameters for R and suppose that

$$x_1 u_1 + \dots + x_{k+1} u_{k+1} = 0$$

is a relation in M . Using the short exact sequence $0 \rightarrow R \xrightarrow{\alpha} M \xrightarrow{\beta} Q \rightarrow 0$, we will build a free presentation of Q as in (2.2), but we want to highlight an element that maps to $\overline{u_{k+1}} \in Q$. Let $F = F_1 \oplus Rf^*$ be a direct sum of free modules that maps onto Q via μ so that $\mu(f^*) = \overline{u_{k+1}}$. (We can just choose F_1 so that it already surjects onto Q and think of f^* as a redundant add-on.) Given this free F and map μ , extend it to a free presentation of Q with a free module G . We can then choose a lift $\tilde{\mu} : F \rightarrow M$ of μ so that $\tilde{\mu}(f^*) = u_{k+1}$ while $\tilde{\mu}$ restricted to F_1 is lifted arbitrarily. Then construct the rest of the maps in (2.7) as in Section 2.

Under these conditions, we study $\text{Im } \nu^\vee$ in G^\vee . Since $F = F_1 \oplus Rf^*$, we have $F^\vee = (F_1)^\vee \oplus (Rf^*)^\vee$. Let $h^* \in (Rf^*)^\vee$ be the map $h^*(f^*) = 1$, and let $y = \nu^\vee(h^*) = h^* \circ \nu \in \text{Im } \nu^\vee$. Then $\text{Im } \nu^\vee = H + Ry$, where $H = \nu^\vee((F_1)^\vee)$.

We can thus construct a commutative diagram as in (2.7) with exact rows with the following extra conditions:

- (1) $F = F_1 \oplus Rf^*$ with $\mu(f^*) = \overline{u_{k+1}}$, $\tilde{\mu}(f^*) = u_{k+1}$, and $\mu_1(f^*, 0) = u_{k+1}$
- (2) $\text{Im } \nu^\vee = H + Ry = \nu^\vee((F_1)^\vee) + R(h^* \circ \nu)$, where $h^*(f^*) = 1$

Lemma 3.2. *Let M be an arbitrary R -module with $\alpha : R \rightarrow M$ an injection. Let x_1, \dots, x_{k+1} be part of a system of parameters for R , $I = (x_1, \dots, x_k)$, and suppose that $x_1 u_1 + \dots + x_{k+1} u_{k+1} = 0$ is a relation in M . Use the construction of Discussion 3.1 applied to diagram (2.7). Let $N = G^\vee/H$. Then there exists a map $\phi : N \rightarrow R/I$ with $\phi(y) = x_{k+1} + I$ and $\tilde{\nu} \in \ker \phi$.*

Proof. Since $x_1 u_1 + \dots + x_{k+1} u_{k+1} = 0$ in M and $\mu_1 : F \oplus R \rightarrow M$ is surjective, there exists $(f_1, r_1), \dots, (f_k, r_k)$ in $F \oplus R$ such that $\mu_1(f_i, r_i) = u_i$ (without loss of generality, we may assume that $f_i \in F_1 \subseteq F$ since F_1 surjects onto Q). Thus,

$$x_1(f_1, r_1) + \dots + x_k(f_k, r_k) + x_{k+1}(f^*, 0) \in \ker \mu_1 = \text{Im } \nu_1.$$

Let $g^* \in G$ be such that

$$\nu_1(g^*) = x_1(f_1, r_1) + \dots + x_k(f_k, r_k) + x_{k+1}(f^*, 0),$$

and define a map $\psi : G^\vee \rightarrow R/I$ by $\psi(h) = h(g^*) + I$. We claim that $\psi(H) \subseteq I$, which would induce a map $\phi : N \rightarrow R/I$. Indeed, if $h \circ \nu \in H = \nu^\vee((F_1)^\vee)$ with $h \in (F_1)^\vee$, then $h \circ \nu(g^*) = h(x_1 f_1 + \dots + x_k f_k + x_{k+1} f^*) \in I$ because $h(f^*) = 0$. Next, $\psi(y) = y(g^*) = h^* \circ \nu(g^*) = h^*(x_1 f_1 + \dots + x_k f_k + x_{k+1} f^*) = x_{k+1} + I$ as $h^*(f_i) = 0$ since $f_i \in F_1$, and $h^*(f^*) = 1$ so that $\phi(y) = x_{k+1} + I$ too. Finally, $\psi(\tilde{\nu}) = \tilde{\nu}(g^*) + I = \alpha^{-1} \circ \tilde{\mu} \circ \nu(g^*) + I = \alpha^{-1} \circ \tilde{\mu}(x_1 f_1 + \dots + x_k f_k + x_{k+1} f^*) + I$, but $\tilde{\mu}(f_i) = \mu_1(f_i, r_i) + \alpha(r_i) = u_i + \alpha(r_i)$ and $\tilde{\mu}(f^*) = u_{k+1}$. Therefore,

$$\psi(\tilde{\nu}) = \alpha^{-1}(x_1 u_1 + \dots + x_k u_k + x_{k+1} u_{k+1} + \alpha(x_1 r_1 + \dots + x_k r_k)) + I = I$$

as $x_1 u_1 + \dots + x_{k+1} u_{k+1} = 0$. So, $\phi(\tilde{\nu}) \in \ker \phi$. \square

Corollary 3.3. *Under the same assumptions and constructions of Lemma 3.2, assume also that cl is a closure operation for R -modules that obeys the big axioms and that $\alpha : R \rightarrow M$ is phantom. Then $\tilde{\nu} \in (H + Iy)_{G^\vee}^{cl}$.*

Proof. Since the extension is phantom, Lemma 2.5 implies that $\tilde{\nu} \in (\text{Im } \nu^\vee)_{G^\vee}^{cl} = (H + Ry)_{G^\vee}^{cl}$. By axiom (4), $\tilde{\nu} + H \in (R\bar{y})_N^{cl}$. Then Lemma 3.2 and Lemma 1.3(c), which requires Axiom (7), implies that $\tilde{\nu} + H \in (I\bar{y})_N^{cl}$. By Lemma 1.3(a), we can pull back to G^\vee to get $\tilde{\nu} \in (H + Iy)_{G^\vee}^{cl}$. \square

Now we are ready to look at a module modification of M which will trivialize the relation $x_1 u_1 + \dots + x_{k+1} u_{k+1} = 0$. Starting with an arbitrary module M , one can produce a balanced big Cohen-Macaulay module through a very large direct limit of modules that are produced by forcing relations on systems of parameters to become trivial eventually. The difficult part is to show that one has not trivialized so much that the end module is just 0. This problem can be avoided by keeping track of a special element that avoids being in $\mathfrak{m}M'$ for all modules M' in the limit. In our case, we will start with an injection $\alpha : R \rightarrow M$ that is phantom and keep track of the image of $\alpha(1)$ throughout. We will keep this element out of $\mathfrak{m}M'$ by showing that all modifications remain phantom extensions of R so that Corollary 2.10 gives the necessary exclusion. See [HH94, Discussion 5.15] for more background on the concept.

Discussion 3.4. Let M be an arbitrary R -module with an injection $\alpha : R \rightarrow M$ and a relation $x_1 u_1 + \cdots + x_{k+1} u_{k+1} = 0$ in M on a partial system of parameters x_1, \dots, x_{k+1} from R . Define the module modification of M by

$$M' := \frac{M \oplus R^k}{R(u_{k+1}, x_1, \dots, x_k)}$$

along with maps $M \rightarrow M'$ given by $u \mapsto \overline{(u, \mathbf{0})}$ and $\alpha' : R \rightarrow M \rightarrow M'$ given by the composition of α and the previous map. Our goal is to prove that α' is a phantom extension when α is phantom. By the proof of [D10, Lemma 3.8], α' is automatically injective.

Now, assume that $\alpha : R \rightarrow M$ is a phantom extension with a free presentation of $Q = M/\alpha(R)$ and commutative diagram as in (2.2). The big task will be to build a commutative diagram similar to (2.2) for $\alpha' : R \rightarrow M'$. Let

$$Q' = M'/\alpha'(R) = \frac{Q \oplus R^k}{R(\overline{u_{k+1}}, x_1, \dots, x_k)}$$

Then $0 \rightarrow R \xrightarrow{\alpha'} M' \xrightarrow{\beta'} Q' \rightarrow 0$ gives a short exact sequence. Using the free presentation $G \xrightarrow{\nu} F \xrightarrow{\mu} Q \rightarrow 0$ of Q from Discussion 3.1 with its special element $f^* \in F$, we will build a free presentation of Q' as $G \oplus R \xrightarrow{\nu'} F \oplus R^k \xrightarrow{\mu'} Q' \rightarrow 0$ via the maps

$$\mu'(f, r_1, \dots, r_k) = \overline{(\mu(f), r_1, \dots, r_k)}$$

and

$$\nu'(g, r) = (\nu(g) + r f^*, r x_1, \dots, r_k x_k) = \nu(g) + r(f^*, x_1, \dots, x_k).$$

We claim that this construction gives a free presentation for Q' . Indeed, since μ is surjective, μ' above will also surject onto Q' . To see exactness in the middle, note that

$$\begin{aligned} \mu' \circ \nu'(g, r) &= \mu'(\nu(g) + r f^*, r x_1, \dots, r_k x_k) \\ &= \overline{(\mu \circ \nu(g) + r \mu(f^*), r x_1, \dots, r_k x_k)} = \overline{(r \overline{u_{k+1}}, r x_1, \dots, r_k x_k)} = 0. \end{aligned}$$

Additionally, if $\mu'(f, r_1, \dots, r_k) = \overline{(\mu(f), r_1, \dots, r_k)} = 0$ in Q' , then within $Q \oplus R^k$ we have

$$(\mu(f), r_1, \dots, r_k) = r(\overline{u_{k+1}}, x_1, \dots, x_k)$$

for some $r \in R$. Hence, $r_i = r x_i$ for all i , and $0 = \mu(f) - r \overline{u_{k+1}} = \mu(f - r f^*)$. This implies that there exists $g \in G$ such that $\nu(g) = f - r f^*$ and so $\nu'(g, r) = (\nu(g) + r f^*, r x_1, \dots, r_k x_k) = (f, r_1, \dots, r_k)$.

We next join the short exact sequence $0 \rightarrow R \xrightarrow{\alpha'} M' \xrightarrow{\beta'} Q' \rightarrow 0$ with the free presentation $G \oplus R \xrightarrow{\nu'} F \oplus R^k \xrightarrow{\mu'} Q' \rightarrow 0$ to form a commutative diagram. Lift μ' to a map $\tilde{\mu}' : F \oplus R^k \rightarrow M'$ such that $\tilde{\mu}'$ restricted to F is just the image of $\tilde{\mu}$, and $\tilde{\mu}'$ restricted to R^k is the same as μ' restricted to R^k . Then lift $\tilde{\mu}' \circ \nu' : G \oplus R \rightarrow \text{Im } \alpha' \subseteq M'$ to $\tilde{\nu}' : G \oplus R \rightarrow R$. We now have a commutative diagram with exact rows

$$(3.5) \quad \begin{array}{ccccccc} R & \xrightarrow{\alpha'} & M' & \xrightarrow{\beta'} & Q' & \longrightarrow & 0 \\ \uparrow \tilde{\nu}' & & \uparrow \tilde{\mu}' & & \uparrow \text{id} & & \\ G \oplus R & \xrightarrow{\nu'} & F \oplus R^k & \xrightarrow{\mu'} & Q' & \longrightarrow & 0 \end{array}$$

In order to prove that $\alpha' : R \rightarrow M'$ is phantom when $\alpha : R \rightarrow M$ is phantom, we need to analyze the condition $\tilde{\nu}' \in (\text{Im } (\nu')^\vee)_{(G \oplus R)^\vee}^{cl}$.

Lemma 3.6. *Using the notation and assumptions of Discussions 3.1 and 3.4, we have the following:*

- (a) $\tilde{\nu}' = \tilde{\nu} \oplus 0 : G \oplus R \rightarrow R$
- (b) $\text{Im } (\nu')^\vee = (H \oplus 0) + R(y \oplus \text{id}) + I(0 \oplus \text{id})$
- (c) *Suppose that cl is a closure operation for R -modules that obeys the big axioms. If $\alpha : R \rightarrow M$ is phantom, then $\alpha' : R \rightarrow M'$ is phantom.*

Proof. For (a), the construction above gives $\tilde{\nu}' = (\alpha')^{-1} \circ \tilde{\mu}' \circ \nu'$ as it does in (2.2). Note that

$$\begin{aligned} \tilde{\mu}' \circ \nu'(g, r) &= \overline{(\tilde{\mu} \circ \nu(g) + r\tilde{\mu}(f^*), rx_1, \dots, rx_k)} \\ &= \overline{(\tilde{\mu} \circ \nu(g), \mathbf{0}) + r(u_{k+1}, x_1, \dots, x_k)} = \overline{(\tilde{\mu} \circ \nu(g), \mathbf{0})}. \end{aligned}$$

Thus,

$$\tilde{\nu}'(g, r) = (\alpha')^{-1} \overline{(\tilde{\mu} \circ \nu(g), \mathbf{0})} = \alpha^{-1} \circ \tilde{\mu} \circ \nu(g) = \tilde{\nu}(g).$$

For (b), write $h \in (F \oplus R^k)^\vee = (F_1 \oplus Rf^* \oplus R^k)^\vee$ as

$$h = h_1 \oplus h_2 \oplus h_3 = h_1 \oplus sh^* \oplus h_3,$$

and write $\nu'(g, r) = \nu(g) + r(f^*, x_1, \dots, x_k)$. Then

$$\begin{aligned} (\nu')^\vee(h)(g, r) &= h \circ \nu'(g, r) \\ &= h_1(\nu(g)) + sh^*(\nu(g)) + rsh^*(f^*) + rh_3(x_1, \dots, x_k) \\ &= h_1(\nu(g)) + s(y(g) + r) + rh_3(x_1, \dots, x_k). \end{aligned}$$

These three terms exactly generate $(H \oplus 0) + R(y \oplus \text{id}) + I(0 \oplus \text{id})$.

For (c), we assume that $\tilde{\nu} \in (\text{Im } \nu^\vee)_{G^\vee}^{cl}$, and we need to show that $\tilde{\nu}' \in (\text{Im } (\nu')^\vee)_{(G \oplus R)^\vee}^{cl}$. By Corollary 3.3 we know that $\tilde{\nu} \in (H + Iy)_{G^\vee}^{cl}$. Thus, $\tilde{\nu}' = \tilde{\nu} \oplus 0 \in (H + Iy)_{G^\vee}^{cl} \oplus 0 \subseteq (H + Iy)_{G^\vee}^{cl} \oplus 0_{R^\vee}^{cl}$ by axiom (1). Then Lemma 1.3(b) implies that $\tilde{\nu}' \in ((H + Iy) \oplus 0)_{(G \oplus R)^\vee}^{cl}$. Finally, axiom (3) achieves the result as $(H + Iy) \oplus 0 \subseteq (H \oplus 0) + R(y \oplus \text{id}) + I(0 \oplus \text{id}) = \text{Im } (\nu')^\vee$. \square

We now have our main theorem of this section, which generalizes the main theorem of [D10] to arbitrary R -modules M , not just finitely generated modules.

Theorem 3.7. *Let R be a local domain possessing an operation cl that obeys the big axioms. If $\alpha : R \rightarrow M$ is a phantom extension, then M can be modified into a balanced big Cohen-Macaulay module over R .*

As an add-on fact, we point out the following lemma regarding modules that factor through phantom extensions.

Lemma 3.8. *Let R be a local domain with a closure operation cl obeying the big axioms, and let M and M_1 be arbitrary R -modules. Suppose that $R \xrightarrow{\alpha_1} M_1 \xrightarrow{\theta} M$ is a sequence of maps such that the composition $\alpha = \theta \circ \alpha_1$ is phantom via cl . Then α_1 is also phantom via cl .*

Proof. Consider the diagram of maps shown below. Below, we will justify that all squares are commutative, the rows are exact, and the vertical paths are also

commutative. One may want to visualize this diagram by rolling it vertically and gluing the top and bottom rows together to produce a commutative prism.

$$(3.9) \quad \begin{array}{ccccccc} & G_1 & \xrightarrow{\nu_1} & F_1 & \xrightarrow{\mu_1} & Q_1 & \longrightarrow 0 \\ & \downarrow \tilde{\nu}_1 & & \downarrow \tilde{\mu}_1 & & \downarrow \text{id} & \\ 0 & \longrightarrow & R & \xrightarrow{\alpha_1} & M_1 & \xrightarrow{\beta_1} & Q_1 \longrightarrow 0 \\ & & \downarrow \text{id} & & \downarrow \theta & & \downarrow \bar{\theta} \\ 0 & \longrightarrow & R & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & Q \longrightarrow 0 \\ & & \uparrow \tilde{\nu} & & \uparrow \tilde{\mu} & & \uparrow \text{id} \\ & G & \xrightarrow{\nu} & F & \xrightarrow{\mu} & Q & \longrightarrow 0 \\ & \uparrow \psi & & \uparrow \phi & & \uparrow \bar{\theta} & \\ & G_1 & \xrightarrow{\nu_1} & F_1 & \xrightarrow{\mu_1} & Q_1 & \longrightarrow 0 \end{array}$$

Start with the maps α_1 and α . As α is injective (because phantom), α_1 is also injective. Use θ to build SESs and commutative squares as in rows 2 and 3 above. Next, choose a free presentation of Q and lifting maps as in row 4. Next, we will build a free presentation for Q_1 , but we need to be careful so that it is compatible with the presentation of Q .

Start with an arbitrary choice of free presentation as in rows 1 and 5 (same modules and maps). We claim that we can lift μ_1 to $\tilde{\mu}_1 : F_1 \rightarrow M_1$ so that $\theta \circ \tilde{\mu}_1 : F_1 \rightarrow \tilde{\mu}(F) \subseteq M$. Indeed, let f_1 be a free basis element for F_1 over R . Then $\mu_1(f_1) = \beta_1(m_1)$ for some $m_1 \in M_1$, and $\bar{\theta} \circ \mu_1(f_1) = \mu(f)$ for some $f \in F$. Therefore,

$$\beta(\tilde{\mu}(f) - \theta(m_1)) = \mu(f) - \bar{\theta} \circ \beta_1(m_1) = \bar{\theta} \circ \mu_1(f_1) - \bar{\theta} \circ \mu_1(f_1) = 0$$

and so $\tilde{\mu}(f) - \theta(m_1) = \alpha(r)$ for some $r \in R$. Define $\tilde{\mu}_1(f_1) = m_1 + \alpha_1(r)$. Then

$$\theta \circ \tilde{\mu}_1(f_1) = \theta(m_1) + \theta \circ \alpha_1(r) = [\tilde{\mu}(f) - \alpha(r)] + \alpha(r) \in \tilde{\mu}(F) \subseteq M$$

Given this definition of $\tilde{\mu}_1$, we lift ν_1 to $\tilde{\nu}_1 : G_1 \rightarrow R$ without extra conditions.

We now define the maps ϕ and ψ to produce commutative squares at the bottom. To define $\phi : F_1 \rightarrow F$ as a lift of μ_1 we note that $\theta \circ \tilde{\mu}_1 : F_1 \rightarrow \tilde{\mu}(F)$ and $\tilde{\mu} : F \rightarrow M$ gives a surjection onto $\tilde{\mu}(F)$. Hence, we can lift $\theta \circ \tilde{\mu}_1$ to a map $\phi : F_1 \rightarrow F$ such that $\theta \circ \tilde{\mu}_1 = \tilde{\mu} \circ \phi$. We can then lift ν_1 to $\psi : G_1 \rightarrow G$ without extra conditions.

To fulfill the requirements for the diagram we need the vertical paths to be commutative. We have already noted that $\theta \circ \tilde{\mu}_1 = \tilde{\mu} \circ \phi$ by construction. Finally,

$$\alpha \circ \tilde{\nu} \circ \psi = \tilde{\mu} \circ \phi \circ \nu_1 = \theta \circ \tilde{\mu}_1 \circ \nu_1 = \alpha \circ \tilde{\nu}_1$$

and α is injective, so that $\tilde{\nu} \circ \psi = \tilde{\nu}_1$.

As a result, we can now prove that α phantom implies that α_1 is phantom. As α is phantom, $\tilde{\nu} \in (\text{Im } \nu^\vee)_{G^\vee}^{cl}$. Applying $\psi^\vee : G^\vee \rightarrow G_1^\vee$, we have

$$\tilde{\nu}_1 = \psi^\vee(\tilde{\nu}) \in (\psi^\vee(\text{Im } \nu^\vee))_{G_1^\vee}^{cl} = (\text{Im}(\nu \circ \psi)^\vee)_{G_1^\vee}^{cl} = (\text{Im}(\phi \circ \nu_1)^\vee)_{G_1^\vee}^{cl} \subseteq (\text{Im } \nu_1^\vee)_{G_1^\vee}^{cl}$$

by using axioms (3) and (4), which shows that α_1 is phantom. \square

4. SOLIDITY AND PHANTOM EXTENSIONS

Hochster [Ho94] defined an R -module M to be solid if $\text{Hom}_R(M, R) \neq 0$. If the module is an R -algebra S , then S is solid if and only if there exists an R -linear map $\gamma : S \rightarrow R$ such that $\gamma(1) = c \neq 0$. While Hochster showed that all balanced big Cohen-Macaulay modules and algebras are solid, an open question from [Ho94] was whether or not all solid algebras can be modified into balanced big Cohen-Macaulay algebras, i.e., are solid algebras also seeds (R -algebras that can be mapped to a balanced big Cohen-Macaulay R -algebra [D07]). In [HH09, Example 3.14] Hochster and Huneke gave a counterexample in characteristic p , which shows that not all solid algebras in characteristic p are seeds even though all seeds are solid.

In this section we provide a complementary result to the connection between solid algebras and seeds in light of the counterexample mentioned above. We will prove that if R is characteristic p and also F -finite (for R reduced, $R \rightarrow R^{1/p}$ is a module-finite extension), then all solid algebras are phantom extensions of R and thus can be modified into big Cohen-Macaulay modules. We will need several characteristic p lemmas to help set up everything.

Lemma 4.1. (cf. [HH94, Proposition 5.12]) *Let R be reduced and have characteristic p with a short exact sequence $0 \rightarrow R \xrightarrow{\alpha} M \xrightarrow{\beta} Q \rightarrow 0$. Then for any $q = p^e$, $0 \rightarrow R^{1/q} \xrightarrow{1 \otimes \alpha} R^{1/q} \otimes_R M \xrightarrow{1 \otimes \beta} R^{1/q} \otimes_R Q \rightarrow 0$ is still a short exact sequence.*

Lemma 4.2. *Let R be a domain of characteristic p . If S is a solid R -algebra, then there exists a fixed $c \neq 0$ in R such that for all $q = p^e$, there is an R -linear map $\gamma_q : R^{1/q} \otimes_R S \rightarrow R^{1/q}$ with $\gamma_q(r^{1/q} \otimes 1) = c^{1/q} r^{1/q}$.*

Proof. Since S is solid, there is an R -linear map $\gamma : S \rightarrow R$ such that $\gamma(1) = c \neq 0$. Using this fixed c (independent of q), we will build the other maps. For each q , consider the R -bilinear map $R^{1/q} \times S \rightarrow R^{1/q}$ given by $(r^{1/q}, s) \mapsto r^{1/q} \gamma(s^q)^{1/q}$. This map induces the required γ_q . \square

Lemma 4.3. *Let R be a Noetherian ring (of any characteristic) with an R -algebra $\rho : R \rightarrow T$ such that T is module-finite over R . Let G be a free (but not necessarily finite) R -module. Then $T \otimes_R \text{Hom}_R(G, R) \cong \text{Hom}_T(T \otimes_R G, T)$.*

Proof. Define $\phi : T \otimes_R \text{Hom}_R(G, R) \rightarrow \text{Hom}_T(T \otimes_R G, T)$ by $\phi(t \otimes h) = \tilde{h}$, where $\tilde{h}(t' \otimes g) = tt' \rho(h(g))$ for all $h \in \text{Hom}_R(G, R)$. To see that ϕ is one-to-one, suppose that $\phi(\sum_i t_i \otimes h_i) = 0$ -map. Then for all $g \in G$, we have $0 = \sum_i \tilde{h}_i(1 \otimes g) = \sum_i t_i \rho(h_i(g))$, which implies that $\sum_i t_i \otimes h_i(g) = \sum_i t_i \rho(h_i(g)) \otimes 1 = 0$. Thus, $\sum_i t_i \otimes h_i$ is also the 0-map in $T \otimes_R \text{Hom}_R(G, R)$. To see that ϕ is onto, let $\{\epsilon_i\}_{i \in \mathcal{I}}$ be a basis for G over R so that $\{1 \otimes \epsilon_i\}_{i \in \mathcal{I}}$ is a basis for $T \otimes_R G$ over T . Let $\tilde{h} \in \text{Hom}_T(T \otimes_R G, T)$ be such that $\tilde{h}(1 \otimes \epsilon_i) = t_i \in T$. Let J be the T -ideal generated by the t_i . As T is module-finite over R and R is Noetherian, J is a finite R -module with generators $\{b_1, \dots, b_n\} \subseteq T$. For all $i \in \mathcal{I}$, write $t_i = \sum_j \rho(r_{ij}) b_j$. Define n maps $h_j : G \rightarrow R$ by $h_j(\epsilon_i) = r_{ij}$. Thus, $\sum_{j=1}^n b_j \otimes h_j$ is in $T \otimes_R \text{Hom}_R(G, R)$, and $\sum_j b_j \otimes h_j(\epsilon_i) = \sum_j b_j \otimes r_{ij} = \sum_j b_j \rho(r_{ij}) \otimes 1 = t_i \otimes 1$. Thus, $\tilde{h} = \phi(\sum_j b_j \otimes h_j)$. \square

Corollary 4.4. *Let R be a reduced, F -finite Noetherian ring of characteristic p , and let G be a free R -module. Then $R^{1/q} \otimes_R \text{Hom}_R(G, R) \cong \text{Hom}_{R^{1/q}}(R^{1/q} \otimes_R G, R^{1/q})$.*

Discussion 4.5. Let (R, \mathfrak{m}) be a local domain, and let S be a solid R -algebra. Then R/I is not solid for any proper ideal I and so $\alpha : R \rightarrow S$ must be an injection. Thus, we have a short exact sequence $0 \rightarrow R \xrightarrow{\alpha} S \xrightarrow{\beta} Q \rightarrow 0$ with an R -linear map $\gamma : S \rightarrow R$ such that $\gamma(1) = c \neq 0$.

If R is also characteristic p , then Lemmas 4.1 and 4.2 imply that for each $q = p^e$, we have a short exact sequence $0 \rightarrow R^{1/q} \xrightarrow{1 \otimes \alpha} R^{1/q} \otimes_R S \xrightarrow{1 \otimes \beta} R^{1/q} \otimes_R Q \rightarrow 0$ with an R -linear map $\gamma_q : R^{1/q} \otimes_R S \rightarrow R$ such that $\gamma_q(r^{1/q} \otimes 1) = c^{1/q} r^{1/q}$, i.e., $\gamma_q \circ (1 \otimes \alpha)$ is multiplication by $c^{1/q}$. So, for each q we can build a diagram

$$(4.6) \quad \begin{array}{ccccccc} R^{1/q} & \xrightarrow{1 \otimes \alpha} & R^{1/q} \otimes_R S & \xrightarrow{1 \otimes \beta} & R^{1/q} \otimes_R Q & \longrightarrow & 0 \\ \uparrow 1 \otimes \tilde{\nu} & & \uparrow 1 \otimes \tilde{\mu} & & \uparrow \text{id} & & \\ R^{1/q} \otimes_R G & \xrightarrow{1 \otimes \nu} & R^{1/q} \otimes_R F & \xrightarrow{1 \otimes \mu} & R^{1/q} \otimes_R Q & \longrightarrow & 0 \end{array}$$

which continues to be commutative with exact rows so that the bottom row is a free presentation of $R^{1/q} \otimes_R Q$ over $R^{1/q}$. Thus,

$$(4.7) \quad c^{1/q}(1 \otimes \tilde{\nu}) = \gamma_q \circ (1 \otimes \alpha) \circ (1 \otimes \tilde{\nu}) = \gamma_q \circ (1 \otimes \tilde{\mu}) \circ (1 \otimes \nu) = (1 \otimes \nu)^\vee (\gamma_q \circ (1 \otimes \tilde{\mu}))$$

and so $c^{1/q} \otimes \tilde{\nu} \in \text{Im}((1 \otimes \nu)^\vee)$ for all q . In other words,

$$c^{1/q} \otimes \tilde{\nu} \in \text{Im}(\text{Hom}_{R^{1/q}}(R^{1/q} \otimes_R F, R^{1/q}) \rightarrow \text{Hom}_{R^{1/q}}(R^{1/q} \otimes_R G, R^{1/q}))$$

but Corollary 4.4 means that for all q

$$c^{1/q} \otimes \tilde{\nu} \in \text{Im}(R^{1/q} \otimes_R \text{Hom}_R(F, R) \rightarrow R^{1/q} \otimes_R \text{Hom}_R(G, R))$$

Hence, $\tilde{\nu} \in (\text{Im } \nu^\vee)_{G^\vee}^*$, the tight closure of $\text{Im } \nu^\vee$ inside G^\vee . Proposition 1.6 shows that the solid algebra S is a phantom extension of R . We therefore arrive at the following theorem.

Theorem 4.8. *Let (R, \mathfrak{m}) be an F -finite, local domain of characteristic p . If S is a solid R -algebra, then S is a phantom extension of R via tight closure.*

Corollary 4.9. *Let (R, \mathfrak{m}) be an F -finite, local domain of characteristic p possessing a big test element (e.g., R is also complete). If S is a solid R -algebra, then S can be modified into a balanced big Cohen-Macaulay module over R .*

Proof. Apply Theorem 3.7 to the fact that tight closure in characteristic p obeys the big axioms in the presence of big test elements. \square

The corollary above stands in contrast to Hochster and Huneke's example in [HH09, Example 3.14] of a solid algebra in characteristic p that cannot be modified into a big Cohen-Macaulay algebra. In the F -finite case, we can now see that although there are solid algebras that are not seeds (cannot be mapped to big Cohen-Macaulay algebras), all solid algebras can be modified into big Cohen-Macaulay modules because solid algebras are phantom extensions. We next point out that if $\alpha : R \rightarrow M$ is a phantom extension over a complete ring R , then M is solid.

Proposition 4.10. *Let (R, \mathfrak{m}) be a complete, local domain. If $\alpha : R \rightarrow M$ is a phantom extension, then M is a solid module.*

Proof. As R is a complete, local domain, it is a module-finite extension of a regular, local domain A . As M is a phantom extension, it can be modified into a balanced big Cohen-Macaulay module B by Theorem 3.7, which is still big Cohen-Macaulay over A . Consider the sequence of maps $f : A \rightarrow R \xrightarrow{\alpha} M \rightarrow B$ sending $1 \mapsto 1 \mapsto \alpha(1) \mapsto u \in B \setminus \mathfrak{m}B$. We claim that this map splits over A , which will show that M is solid over A and thus solid over R by [Ho94, Corollary 2.3]. To see that the map $f : A \rightarrow B$ above splits, first let x_1, \dots, x_d be a full, regular system of parameters for A , and let $I_t = (x_1^t, \dots, x_d^t)$ and $y_t = (x_1 \cdots x_d)^t$ for all $t \in \mathbb{N}$. Then $\varinjlim A/I_t = H_{\mathfrak{m}}^d(A) \cong E$, where the maps in the direct limit are multiplication by y_1 and E is the injective hull of the residue field for A . Consider the maps $A/I_t \rightarrow B/I_t B$ induced by f . Then the image of y_{t-1} in A/I_t maps to a nonzero element in $B/I_t B$ since B is big Cohen-Macaulay over A , which shows that in the direct limit $E = A \otimes_A E \xrightarrow{f \otimes 1} B \otimes_A E$ is injective. Therefore, $\text{Hom}_A(B \otimes_A E, E) \rightarrow \text{Hom}_A(E, E)$ is surjective. By adjointness of tensor and Hom , we have $\text{Hom}_A(B, \text{Hom}_A(E, E)) \rightarrow \text{Hom}_A(E, E)$, and by Matlis duality we have $\text{Hom}_A(B, A) \rightarrow A$ is surjective, where the action of this map is $g \mapsto g(f(1))$. Let $\theta \in \text{Hom}_A(B, A)$ be the map that goes to $1 \in A$. Then $\theta(f(1)) = 1$, which gives the splitting map for f . \square

While phantom extensions yield solid modules, not all solid modules are phantom extensions as the example below shows.

Example 4.11. Let R be a regular local domain of characteristic p and dimension at least 2. Let I be any ideal minimally generated by at least 2 elements, and let $\alpha : R \rightarrow I$ be an injection with 1 mapping to one of the minimal generators of I . The inclusion $I \subseteq R$ shows that I is a solid module over R . Moreover, we claim that I is not a phantom extension via tight closure. Indeed, α is not a split map as $I = \alpha(R) \oplus W$ implies that W must be a torsion module since $I \subseteq R$ is a rank 1 torsion-free submodule. However, this would imply that $W = 0$, i.e., $I = \alpha(R)$, which is impossible since I is minimally generated by at least 2 elements. Since R is regular, it is also weakly F -regular. If $\alpha : I \rightarrow R$ were a phantom extension, then the corresponding diagram (2.2) would show that $\tilde{\nu} \in (\text{Im } \nu^\vee)_G^* = \text{Im } \nu^\vee$. Finally, $\tilde{\nu} = h \circ \nu$ for some $h : F \rightarrow R$ implies that we can define $\theta : Q \rightarrow I$ by $\theta(q) = \tilde{\mu}(f) - \alpha \circ h(f)$, where $f \in F$ such that $\mu(f) = q$. (The commutativity of (2.2) shows that γ is well-defined and independent of the choice of f .) Since $\beta \circ \theta(q) = \beta \circ \tilde{\mu}(f) - \beta \circ \alpha \circ h(f) = \mu(f) - 0 = q$ for all $q \in Q$, we see that β (and so α) would be split maps. Therefore, I is a solid R -module that is not a phantom extension of R .

5. OPEN QUESTIONS

We close with some open questions that are of great interest for later study.

- (1) Suppose that $\alpha : R \rightarrow B$ is a map to a big Cohen-Macaulay module B . For which maps, if any, is α a phantom extension?
- (2) Are solid algebra maps $\alpha : R \rightarrow S$ always phantom extensions? In other words, can we remove the F -finite (or characteristic p) hypotheses of Theorem 4.8 using other characteristic p methods or by only employing the characteristic-free axioms?
- (3) Can either of the above questions be answered by adding new axioms to the list presented in this article? Any new axiom should follow from the existence

of big Cohen-Macaulay modules and preferably also be a property held by tight closure.

- (4) Can one define closure operation axioms that lead to big Cohen-Macaulay algebras? In an upcoming paper [RG], Rebecca R.G. has produced a positive answer to the this question when one adds an additional axiom involving preservation of a phantom extension after passing to a second symmetric power of a module.

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